STABILITY ESTIMATE FOR A PARTIAL DATA INVERSE PROBLEM FOR THE CONVECTION-DIFFUSION EQUATION

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Abstract. In this article, we study the stability in the inverse problem of determining the time-dependent convection term and density coefficient appearing in the convection-diffusion equation, from partial boundary measurements. For dimension $n \geq 2$, we show the convection term (modulo the gauge term) admits log-log stability, whereas log-log-log stability estimate is obtained for the density coefficient.

Keywords: Inverse problems, partial Dirichlet to Neumann map, parabolic equation, Carleman estimates, stability estimate.

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1. Introduction

For $n \geq 2$, let $\Omega \subset \mathbb{R}^n$ be a bounded simply connected domain having C^2 smooth boundary $\Gamma = \partial\Omega$. For $T > 0$, let us introduce the parabolic operator $\mathcal{L}_{A,q}$ in the cylinder $Q := (0,T) \times \Omega$ by

$$\mathcal{L}_{A,q} := \partial_t - \sum_{j=1}^{n} \left(\partial_j + A_j(t,x)\right)^2 + q(t,x), \tag{1.1}$$

where $A(t,x) := (A_1(t,x), A_2(t,x), ..., A_n(t,x)) \in (W^{1,\infty}(Q))^n$ and $q \in L^\infty(Q)$. We consider an initial boundary value problem (IBVP) known for the convection-diffusion equation which models physical processes like mass or heat transfer within a body and, also appears in probabilistic study of diffusion process (like, the Fokker-Planck and Kolmogorov equations), finance (like, the BlackScholes or the Ornstein-Uhlenbeck processes) and chemical engineering (for describing the movement of macro-particles)

$$\begin{cases} \mathcal{L}_{A,q}u(t,x) = 0, & (t,x) \in Q, \\ u(0,x) = 0, & x \in \Omega, \\ u(t,x) = f(t,x), & (t,x) \in \Sigma := (0,T) \times \Gamma. \end{cases} \tag{1.2}$$

We first briefly discuss some well-posedness results regarding the above IBVP. Following [17], one can consider suitable spaces for the forward problem (1.2). However, in the context of our article, we assume more regularity on the coefficients in the operator (1.1). Inspired by [6], for a given $m > 0$, we define the admissible set $\mathcal{M}(m)$ of coefficients A and q by

$$\mathcal{M}(m) = \left\{(A,q) \in H^k(Q;\mathbb{R}^n) \times H^{k-1}(Q;\mathbb{R}); \|A\|_{H^k(Q;\mathbb{R}^n)} + \|q\|_{H^{k-1}(Q;\mathbb{R})} \leq m \text{ and } k > \tfrac{1+n}{2} + 3\right\}.$$

Now, we introduce some time-dependent Sobolev spaces for p, q being non-negative real numbers and $M = \Omega$ or, Γ

$$H^{p,q}\left((0,T) \times M\right) := L^2\left(0,T; H^p(M)\right) \cap H^q\left(0,T; L^2(M)\right),$$

equipped with the norm

$$\|u\|_{H^{p,q}((0,T)\times M)} = \|u\|_{L^2(0,T;H^p(M))} + \|u\|_{H^q(0,T;L^2(M))}.$$

Further, we denote $H_0^{p,q}(\Sigma) := \{f \in H^{p,q}(\Sigma); f(0,x) = 0,\ x \in \Omega\}$. The existence of unique solution $u \in H^{2,1}(Q)$ to the IBVP (1.2) for a given Dirichlet data $f \in H_0^{\frac{3}{2},\frac{3}{4}}(\Sigma)$ follows from [47]. Also then, we have some $C > 0$ depending only on m and Q such that

$$\|u\|_{H^{2,1}(Q)} \leq C\|f\|_{H_0^{\frac{3}{2},\frac{3}{4}}(\Sigma)}.$$

We can define the Dirichlet to Neumann (DN) map $\Lambda^*_{A,q} : H_0^{\frac{3}{2},\frac{3}{4}}(\Sigma) \to H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$ as

$$\Lambda^*_{A,q}(f) = (\partial_\nu u + 2(\nu \cdot A)u)\,|_{\Sigma},$$

where u solves (1.2) and $\nu(x)$ denotes the unit outward normal at $x \in \Gamma$.

The inverse problem under consideration here is the stable recovery of time-evolving properties of a homogeneous medium such as A and q, by applying heat source on Σ and measuring the heat flux on a part of Σ. To be precise, we study the stability aspects for unique recovery of (A, q) from a partial DN map which measures the Neumann outputs on a part of Σ related to a small open neighborhood of the ω_0-illuminated face which is defined in (2.1). It is known that the convection term can be recovered only under the divergence free condition (with respect to space variables) because of the non-uniqueness associated to the gauge transform (see [51, 55]). We derive a stability estimate for the divergence free convection term A. In doing so, a stability result for analytic continuation from [49] will be very helpful (see also [57]). Also we borrow an important construction for the principal term in the geometric optics solutions from [43] which was originally used in the framework of dynamical Schrödinger equation. The decay in the remainder terms of the geometric optics solutions follows from a Carleman estimate. For stability of the density coefficient, we again use the stability result for analytic coninuation in combination with the stability estimate of A.

The issues regarding unique and stable determination of coefficients appearing in parabolic PDEs from boundary measurements have attracted much attention during last several decades. Motivated by the seminal work [56] by Sylvester and Uhlmann, Isakov in [32] uniquely determined the time-dependent coefficient when $A = 0$, by using an argument based on completeness of the product of solutions. The stability issues of the same problem has been resolved by Choulli in [23]. In [2], Avdonin and Seidman used boundary control (BC) method pioneered by Belishev which is further developed by Katchalov, Kurylev and Lassas (see [4, 35] and references therein), to establish uniqueness result for time-independent q. In the absence of any zeroth order term, Cheng and Yamamoto proved in [18, 19, 20] uniqueness of convection term which belongs to some Lebesgue spaces from single measurement when $n = 2$. Gaitan and Kian in [30] obtained stable determination result for time-dependent q in a bounded cylindrical domain when $A = 0$ which was further generalized in the article [45] by Kian and Yamamoto proving analogous results in time-fractional diffusion equation settings. In [25], Choulli and Kian derived logarithmic stability estimates for time-dependent term q working only with partial DN map, in the absence of first order coefficients. In [12], Bellassoued and Rassas stably determined the convection term A and density coefficient q both of which are time-independent. Vashisth and Sahoo in [55] obtained unique determination result for time-dependent convection term (modulo gauge equivalence) and density coefficient from full Dirichlet and partial Neumann data. In this work, we have proved the stability estimate for determining the time-dependent convection term and the density coefficients from the knowledge of full Dirichlet data and the Neumann data measured on a portion which is slightly bigger than half of the lateral boundary. We refer to [10, 21, 22, 24, 26, 23, 27, 30, 33, 34, 50] for more works in inverse problems related to parabolic PDEs. Also, there have been a considerable amount of work done in the context of hyperbolic and dynamical Schrödinger equations (see [1, 3, 7, 11, 14, 15, 29, 41, 42, 43, 44, 13, 52, 53, 8, 9, 31, 39, 37, 38, 40, 46, 54, 48, 36] and references therein).

The article is organized as follows. In §2, the main result of the article is stated. Then boundary and interior Carleman estimates for the operator $\mathcal{L}_{A,q}$ are derived in §3, followed by the construction of geometric optics solutions in §4. Finally, we discuss the stable determination results for the convection term A and density coefficient q in §5.

2. Statement of the main result

We begin this section by introducing some notations. Following [16], fix an $\omega_0 \in \mathbb{S}^{n-1}$ and define the ω_0-shadowed and ω_0-illuminated faces by

$$\partial\Omega_+(\omega_0) := \{x \in \partial\Omega :\ \nu(x)\cdot\omega_0 \geq 0\}\,, \quad \partial\Omega_-(\omega_0) := \{x \in \partial\Omega :\ \nu(x)\cdot\omega_0 \leq 0\}$$

of $\partial\Omega$ where $\nu(x)$ is the outward unit normal to $\partial\Omega$ at $x \in \partial\Omega$. For a given small $\epsilon > 0$, we define the small open neighborhoods of $\partial\Omega_+(\omega_0)$ and $\partial\Omega_-(\omega_0)$ by

$$\partial\Omega_{+,\epsilon/2}(\omega_0) := \left\{x \in \partial\Omega;\ \nu(x)\cdot\omega_0 > \frac{\epsilon}{2}\right\}, \quad \text{and} \quad \partial\Omega_{-,\epsilon/2}(\omega_0) := \left\{x \in \partial\Omega;\ \nu(x)\cdot\omega_0 < \frac{\epsilon}{2}\right\}. \tag{2.1}$$

respectively. Corresponding to $\partial\Omega_\pm(\omega_0)$ and $\partial\Omega_{\pm,\epsilon/2}(\omega_0)$, we denote the lateral boundary parts by $\Sigma_\pm(\omega_0) := (0,T)\times\partial\Omega_\pm(\omega_0)$ and $\Sigma_{\pm,\epsilon/2}(\omega_0) := (0,T)\times\partial\Omega_{\pm,\epsilon/2}(\omega_0)$ respectively. Let us define the partial Dirichlet to Neumann map denoted by $\Lambda_{A,q} : H_0^{\frac{3}{2},\frac{3}{4}}(\Sigma) \to H^{\frac{1}{2},\frac{1}{4}}(\Sigma_{-,\epsilon/2}(\omega_0))$ as

$$\Lambda_{A,q}(f) = (\partial_\nu u + 2(A\cdot\nu)u)\big|_{\Sigma_{-,\epsilon/2}(\omega_0)} \tag{2.2}$$

We now state the main result of this article.

Theorem 2.1. *Let $(A_i, q_i) \in \mathcal{M}(m),\ i = 1,2$ and $T > \operatorname{diam}\,\Omega$, where $\Omega \subset \mathbb{R}^n$ is a bounded C^2 smooth domain for $n \geq 2$. We denote by Λ_i the partial DN map corresponding to $\mathcal{L}_{A_i,q_i}$ as defined in (2.2). Under the assumption $A_1|_\Sigma = A_2|_\Sigma$ and $\nabla_x\cdot A_1 = \nabla_x\cdot A_2$ in Q, we have the following estimates for some positive constants $C, \alpha_1, \alpha_2, \beta_1$ and β_2 depending on m and Q*

$$\begin{aligned}\|A_1 - A_2\|_{L^2(Q)} &\leq C\left(\|\Lambda_1 - \Lambda_2\|^{\alpha_1} + |\log|\log\|\Lambda_1 - \Lambda_2\|||^{\alpha_2}\right),\\ \|q_1 - q_2\|_{L^2(Q)} &\leq C\left(\|\Lambda_1 - \Lambda_2\|^{\beta_1} + |\log|\log|\log\|\Lambda_1 - \Lambda_2\|||\,||^{\beta_2}\right).\end{aligned}$$

We remark here that, in the recent work [6], Bellassoued and Ben Fraj discussed the stability aspects of determining the time-dependent coefficients appearing in the convection-diffusion equation and proved logarithmic and double logarithmic stability results for the convection term and density coefficient respectively. Moreover the Neumann measurements there are taken on any arbitrary part of the lateral boundary Σ; but the coefficients in [6] are assumed to be known in an open set containing Σ which is essential to apply a local unique continuation result near the boundary. In contrast to [6], we consider coefficients which agree only on the lateral boundary. Although we work with the Neumann data measured on a particular subset of Σ, which is slightly more than half of the boundary and obtain double and triple logarithmic stability estimates for A and q respectively.

3. Boundary and interior Carleman estimates

We prove here a boundary Carleman estimate involving for the operator $\mathcal{L}_{A,q}$ which will be used to control the boundary terms appearing in the integral identity given by (5.6) where no information is given. Now, we choose $x_0 \in \mathbb{R}^n$ such that $\inf_{x\in\overline{\Omega}} (x+x_0)\cdot\omega > 0$. For this choice of ω and x_0, the derivation of Carleman estimate goes as follows.

Theorem 3.1. *Let $\phi(t,x) = \lambda^2 t + \lambda x\cdot\omega$ and $u \in C^2(\overline{Q})$ with $u(0,\cdot) = 0$ and $u|_\Sigma = 0$. For $(A,q) \in \mathcal{M}(m)$ there exist $\lambda_1, C > 0$, depending only on m and Q such that*

$$\begin{aligned}\int_Q e^{-2\phi(t,x)}\left(\lambda^2|u(t,x)|^2 + |\nabla_x u(t,x)|^2\right)\,\mathrm{d}x\mathrm{d}t + \int_\Omega e^{-2\phi(T,x)}\left(\lambda|u(T,x)|^2 + |\nabla_x u(T,x)|^2\right)\,\mathrm{d}x\\ + \lambda\int_{\Sigma_{+,\omega}} e^{-2\phi(t,x)}\omega\cdot\nu(x)|\partial_\nu u(t,x)|^2\,\mathrm{d}S_x\mathrm{d}t \le C\int_Q e^{-2\phi(t,x)}|\mathcal{L}_{A,q}u(t,x)|^2\,\mathrm{d}x\mathrm{d}t\\ + C\lambda\int_{\Sigma_{-,\omega}} e^{-2\phi(t,x)}|\omega\cdot\nu(x)||\partial_\nu u(t,x)|^2\,\mathrm{d}S_x\mathrm{d}t\end{aligned} \tag{3.1}$$

holds for all $\lambda \ge \lambda_1$.

Proof. We have to convexify the Carleman weight ϕ appropriately due to the presence of first order derivatives in $\mathcal{L}_{A,q}$. For a proof of the boundary Carleman estimate, we refer to [17] where the following convexified weight has been considered for $s > 0$

$$\phi_s(t,x) := \lambda^2 t + \lambda x\cdot\omega - \frac{s\left((x+x_0)\cdot\omega\right)^2}{2}$$

where $x_0 \in \mathbb{R}^n$ is as mentioned in the line just before the statement of Theorm 3.1. □

We write down the interior Carleman estimate which easily follows from Theorem 3.1 and will be used to construct the geometric optics solutions.

Corollary 3.2. (Interior Carleman estimate). *For $(A,q) \in \mathcal{M}(m)$ there exist $\lambda_1, C > 0$ depending only on m and Q such that the following estimate holds for $u \in C_c^\infty(Q)$ and $\lambda \ge \lambda_1$*

$$\int_Q e^{-2\phi(t,x)}\left(\lambda^2|u(t,x)|^2 + |\nabla_x u(t,x)|^2\right)\,\mathrm{d}x\mathrm{d}t \le C\int_Q e^{-2\phi(t,x)}|\mathcal{L}_{A,q}u(t,x)|^2\,\mathrm{d}x\mathrm{d}t.$$

4. Construction of geometric optics solutions

In this section, we construct the geometric optics solutions for the parabolic operator $\mathcal{L}_{A,q}$ and its formal L^2 adjoint $\mathcal{L}^*_{A,q} = \mathcal{L}_{-A,\overline{q}}$. For $\lambda > 0$, let $\phi(t,x) := \lambda^2 t + \lambda x\cdot\omega$ be the weight function. Then we construct the geometric optics solutions u and v for the operators $\mathcal{L}_{A,q}$ and $\mathcal{L}^*_{A,q}$ respectively which have the following forms

$$\begin{aligned} u(t,x) &= e^{\phi(t,x)}\left(B_g + R_g\right)(t,x),\\ \text{and}\quad v(t,x) &= e^{-\phi(t,x)}\left(B_d + R_d\right)(t,x).\end{aligned} \tag{4.1}$$

Next we show that for λ large enough, the remainder terms R_g and R_d can be estimated in terms of their principal terms B_g and B_d respectively. The decay of R_d and R_g in λ will be crucial to derive stability results from an integral identity obtained by using the solution to an adjoint problem and the given data.

We start with some definition and notations. For $m \in \mathbb{R}$, we define $L^2(0,T;H^m_\lambda(\mathbb{R}^n))$ by

$$L^2(0,T;H^m_\lambda(\mathbb{R}^n)) := \{u(t,\cdot) \in \mathcal{S}'(\mathbb{R}^n) : \left(\lambda^2+|\xi|^2\right)^{m/2} \mathcal{F}_x u(t,\xi) \in L^2((0,T)\times\mathbb{R}^n)\},$$

equipped with the norm

$$\|u\|^2_{L^2(0,T;H^m_\lambda(\mathbb{R}^n))} = \int_0^T \int_{\mathbb{R}^n} \left(\lambda^2+|\xi|^2\right)^m |\mathcal{F}_x u(t,\xi)|^2 \, \mathrm{d}\xi dt,$$

where $\mathcal{S}'(\mathbb{R}^n)$ denote the space of tempered distributions on $\mathbb{R}^n$ and $\mathcal{F}_x$ is the Fourier transform with respect to the space variables. For $0<\delta<<1$, we consider a sequence $\eta_\delta \in C^\infty_c(\delta, T-\delta)$ such that

$$\eta_\delta \equiv 1 \text{ on } [2\delta, T-2\delta] \text{ and, } \quad \|\eta_\delta\|_{W^{k,\infty}(\mathbb{R})} \le C\delta^{-k}, \quad \text{for } k\in\mathbb{N}.$$

Theorem 4.1. *Let $\delta \in (0,T/4)$, $\mathcal{L}_{A,q}$ be as in (1.1) and for $\omega\in\mathbb{S}^{n-1}$, let $\phi(t,x) = \lambda^2 t + \lambda x\cdot\omega$.*

(1) (Exponentially growing solutions) For $(\tau,\xi)\in\mathbb{R}^{1+n}$ such that $\omega\cdot\xi = 0$, $(A,q)\in\mathcal{M}(m)$ and $D\in W^{3,\infty}(Q;\mathbb{R}^n)$ with $\|D\|_{W^{3,\infty}(Q;\mathbb{R}^n)} \le C_0$, there exists $\lambda_0>0$ depending on m, C_0 and Q such that for $\lambda\ge\lambda_0$, we can find $v_g \in H^{2,1}((0,T)\times\Omega)$ solution to

$$\begin{cases} \mathcal{L}_{A,q} v(t,x) = 0, & (t,x)\in Q,\\ v(0,x) = 0, & x\in\Omega \end{cases}$$

taking the form

$$v_g(t,x) = e^{\phi(t,x)}\left(B_g(t,x) + R_g(t,x)\right) \tag{4.2}$$

where $B_g(t,x)$ is given by

$$B_g(t,x) = \eta_\delta(t)\frac{\xi}{|\xi|}\cdot\nabla_x\left(e^{-i(\tau,\xi)\cdot(t,x)} e^{\left(\int_{\mathbb{R}}\omega\cdot D(t,x+s\omega)\,\mathrm{d}s\right)}\right) e^{\left(\int_0^\infty \omega\cdot A(t,x+s\omega)\,\mathrm{d}s\right)} \tag{4.3}$$

and R_g satisfies the following estimate

$$\|R_g\|_{L^2(0,T;H^k(\Omega))} \le C\lambda^{-1+k}\delta^{-3}\langle\tau,\xi\rangle^3, \quad \text{for } k\in\{0,1,2\}. \tag{4.4}$$

(2) (Exponentially decaying solutions) For $(\mathcal{A},q)\in\mathcal{M}(m)$, there exists $\lambda_0>0$ depending on m and Q such that for $\lambda\ge\lambda_0$, we can find $v_d\in H^{2,1}((0,T)\times\Omega)$ solution to

$$\begin{cases} \mathcal{L}^*_{A,q} v(t,x) = 0, & (t,x)\in Q,\\ v(T,x) = 0, & x\in\Omega. \end{cases}$$

taking the form

$$v_d(t,x) = e^{-\phi(t,x)}\left(B_d(t,x) + R_d(t,x)\right) \tag{4.5}$$

where $B_d(t,x)$ is given by

$$B_d(t,x) = \eta_\delta(t) e^{\left(\int_0^\infty \omega\cdot A(t,x+s\omega)\,\mathrm{d}s\right)} \tag{4.6}$$

and R_d satisfies the following estimates

$$\|R_d\|_{L^2(0,T;H^k(\Omega))} \le C\lambda^{-1+k}\delta^{-3}, \quad \text{for } k\in\{0,1,2\}.$$

The proof of the above Theorem relies mainly on some arguments from functional analysis as we need to consider appropriate functional which would be extended and identified by Hahn-Banach and Riesz representation theorems. But continuity of such functional would be possible once we obtain suitable negative order Carleman estimates. Thus we state the following Carleman estimate in a negative order Sobolev space and proof of this estimate follows from very standard arguments (see [6, 8, 9, 17, 28] and references therein). To state the Carleman estimate, we define the conjugated operator $\mathcal{P}_\lambda$ by

$$\mathcal{P}_\lambda := e^{-\phi}\mathcal{L}_{A,q}e^{\phi}.$$

Proposition 4.2. (Shifting the index). *Let A, q, ϕ and $\mathcal{L}_{A,q}$ be as in Theorem 4.1.*

(1) Let $\mathcal{P}_\lambda := e^{-\phi}\mathcal{L}_{A,q}e^{\phi}$, then there exists λ_0 and $C > 0$ such that for $u \in C^1([0,T]; C_c^\infty(\Omega))$ with $u(T,\cdot) = 0$ and $\lambda \geq \lambda_0$, we have

$$\|u\|_{L^2(0,T;H_\lambda^{-1}(\mathbb{R}^n))} \leq C\|\mathcal{P}_\lambda u\|_{L^2(0,T;H_\lambda^{-2}(\mathbb{R}^n))}. \tag{4.7}$$

(2) Let $\mathcal{P}_\lambda^* := e^{\phi}\mathcal{L}_{A,q}^* e^{-\phi}$, then there exists λ_0 and $C > 0$ such that for $u \in C^1([0,T]; C_c^\infty(\Omega))$ with $u(0,\cdot) = 0$ and $\lambda \geq \lambda_0$, we have

$$\|u\|_{L^2(0,T;H_\lambda^{-1}(\mathbb{R}^n))} \leq C\|\mathcal{P}_\lambda^* u\|_{L^2(0,T;H_\lambda^{-2}(\mathbb{R}^n))}. \tag{4.8}$$

For the sake of completeness, we prove the following standard proposition which will lead to the proof of Theorem 4.1.

Proposition 4.3. *For $f \in L^2(0,T; H^1(\mathbb{R}^n))$ there exists $w \in H^{2,1}((0,T)\times\mathbb{R}^n)$ solving the IVP*

$$\begin{cases} \mathcal{P}_\lambda w = f, & \text{in } Q, \\ w(0,x) = 0, & x \in \Omega \end{cases}$$

satisfying, $\|w\|_{L^2(0,T;H_\lambda^2(\Omega))} \leq C\|f\|_{L^2(0,T;H_\lambda^1(\Omega))}$ for some $C > 0$ depending only on m and Q.

Proof. Consider the space $\mathcal{W} := \{\mathcal{P}_\lambda^* u : u \in C^1([0,T]; C_c^\infty(\Omega)) \text{ and } u(T,\cdot) = 0\}$ equipped with the norm $\|\cdot\|_{L^2(0,T;H_\lambda^{-2}(\mathbb{R}^n))}$. Now for $f \in L^2(0,T; H^1(\mathbb{R}^n))$, define a functional T_f on $\mathcal{W}$ by

$$T_f(\mathcal{P}_\lambda^* u) := \int_{\mathbb{R}^{1+n}} f(t,x)u(t,x)\,\mathrm{d}x\mathrm{d}t. \tag{4.9}$$

Now using (4.7), we have that T_f is a continuous linear functional on $\mathcal{W}$ with

$$\|T_f\|_{\mathcal{W}} \leq C\|f\|_{L^2(0,T;H_\lambda^1(\mathbb{R}^n))} \tag{4.10}$$

and hence by the Hahn-Banach Theorem, T_f can be extended to $L^2(0,T; H_\lambda^{-2}(\mathbb{R}^n))$ which will be still denoted by T_f and satisfy (4.10). Finally using the Riesz representation theorem, there exists $w \in L^2(0,T; H_\lambda^2(\mathbb{R}^n))$ such that for $v \in L^2(0,T; H_\lambda^{-2}(\mathbb{R}^n))$ we have

$$T_f(v) = \int_{\mathbb{R}^{1+n}} v(t,x)w(t,x)\,\mathrm{d}x\mathrm{d}t. \tag{4.11}$$

Now combining (4.9) and (4.11) we obtain

$$\int_{\mathbb{R}^{1+n}} \mathcal{P}_\lambda^* v(t,x)w(t,x)\,\mathrm{d}x\mathrm{d}t = \int_{\mathbb{R}^{1+n}} v(t,x)f(t,x)\,\mathrm{d}x\mathrm{d}t, \tag{4.12}$$

for $v \in C^1([0,T]; C_c^\infty(\Omega))$ such that $v(T,\cdot) = 0$ in Ω. This gives us $\mathcal{P}_\lambda w = f$ in Q. Since $f \in L^2(0,T; H^1(\mathbb{R}^n))$ we have $w \in H^1(0,T; L^2(\mathbb{R}^n))$. Now for $v \in C^1([0,T]; C_c^\infty(\Omega))$ satisfying $v(T,\cdot) = 0$, we use the integration by parts to (4.12) to obtain

$$\int_\Omega w(0,x)v(0,x)\,\mathrm{d}x = 0.$$

Hence $w(0,\cdot) = 0$ in Ω. Finally we use (4.10) and (4.11) to get

$$\|T_f\|_{L^2(0,T;H_\lambda^{-2}(\Omega))} = \|w\|_{L^2(0,T;H_\lambda^2(\Omega))} \leq C\|f\|_{L^2(0,T;H_\lambda^1(\Omega))}.$$

$\square$

4.1. **Proof of Theorem 4.1.** First observe that

$$\mathcal{L}_{A,q}\left(e^\phi v\right) = e^\phi\left(\mathcal{L}_{A,q}v - 2\lambda\omega\cdot(\nabla_x + A)v\right).$$

Now using the expressions for v_g, B_g and $\mathcal{L}_{A,q}v_g = 0$, we have that the remainder terms R_g solves

$$\mathcal{P}_\lambda R_g = -\mathcal{L}_{A,q}B_g \tag{4.13}$$

where $\mathcal{P}_\lambda := e^{-\phi}\mathcal{L}_{A,q}e^\phi$ is the conjugated operator as defined earlier and B_g solves the following transport equation

$$\omega\cdot(\nabla_x + A)\,B_g = 0.$$

Hence from (4.13) and Proposition 4.3, it is clear that the remainder term R_g satisfies

$$\|R_g\|_{L^2(0,T;H^k(\Omega))} \leq C\lambda^{-1+k}\|B_g\|_{H^3(Q)},\ \text{for } k\in\{0,1,2\}.$$

This completes the proof for the construction of exponentially growing solutions to $\mathcal{L}_{A,q}v = 0$. One can carry out exactly same set of arguments to prove the existence of exponentially decaying solutions having the form given by equation (4.5) and solution to $\mathcal{L}_{A,q}^*v = 0$. This complete the proof of Theorem 4.1. Hence from (4.13) and Proposition 4.3, it is clear that the remainder term R_g satisfies

$$\|R_g\|_{L^2(0,T;H^k(\Omega))} \leq C\lambda^{-1+k}\|B_g\|_{H^3(Q)},\ \text{for } k\in\{0,1,2\}.$$

This completes the proof for the construction of exponentially growing solutions to $\mathcal{L}_{A,q}v = 0$. One can carry out exactly same set of arguments to prove the existence of exponentially decaying solutions having the form given by equation (4.5) and solution to $\mathcal{L}_{A,q}^*v = 0$. This complete the proof of Theorem 4.1.

5. Proof of theorem 2.1

In this section, we prove the main result on stability for the first and zeroth order coefficients. But first we derive an integral identity using Green's formula where we will plug in the geometric optics solutions constructed before. We simultaneously consider exponentially growing and decaying solutions to avoid any exponential term or boundary terms at initial or final time. To be precise, we construct u_2 and v as the exponentially growing and decaying solutions for the operators $\mathcal{L}_{A_2,q_2}$ and $\mathcal{L}_{-A_1,\overline{q}_1}$ respectively by using Theorem 4.1. Taking $0 < \delta << 1$, $(\tau,\xi) \in \mathbb{R}^{1+n}$ with $\xi\cdot\omega = 0$ and $D(t,x) = A(t,x) := (A_1 - A_2)(t,x)$ we have

$$u_2(t,x) = e^{\phi(t,x)}\left(B_2 + R_2\right)(t,x), \tag{5.1}$$

$$\text{and}\quad v(t,x) = e^{-\phi(t,x)}\left(B + R\right)(t,x) \tag{5.2}$$

where

$$B_2(t,x) = \eta_\delta(t)\frac{\xi}{|\xi|}\cdot\nabla_x\left(e^{-i(t\tau+x\cdot\xi)}e^{\left(\int_{\mathbb{R}}\omega\cdot A(t,x+s\omega)\ \mathrm{d}s\right)}\right)e^{\left(\int_0^\infty \omega\cdot A_2(t,x+s\omega)\ \mathrm{d}s\right)},$$
$$B(t,x) = \eta_\delta(t)e^{\left(-\int_0^\infty \omega\cdot A_1(t,x+s\omega)\ \mathrm{d}s\right)}$$

and $R_2, R \in L^2(0,T;H^2_\lambda(\Omega))$ satisfy

$$\|R_2\|_{L^2(0,T;H^k_\lambda(\Omega))} \leq C\lambda^{-1+k}\delta^{-3}\langle\tau,\xi\rangle^3 \text{ and } \|R\|_{L^2(0,T;H^k_\lambda(\Omega))} \leq C\lambda^{-1+k}\delta^{-3},\ \text{for } k\in\{0,1,2\}. \tag{5.3}$$

Hence, we have for some $\beta > 0$

$$\|u_2\|_{L^2(\Sigma)} \leq e^{\beta\lambda}\delta^{-3}\langle\tau,\xi\rangle^3. \tag{5.4}$$

Also, observe that $u_2(0,x) = v(T,x) = 0$ for $x\in\Omega$. Now, consider u_1 to be the solution of the IBVP

$$\begin{cases}\mathcal{L}_{A_1,q_1}w(t,x) = 0, & (t,x)\in Q,\\ w(0,x) = 0, & x\in\Omega\\ w(t,x) = u_2(t,x), & (t,x)\in\Sigma\end{cases}$$

and define $u := u_1 - u_2$ in Q. Then we get

$$\begin{cases}\mathcal{L}_{A_1,q_1}u(t,x) = 2A(t,x)\cdot\nabla_x u_2(t,x) + \widetilde{q}(t,x)u_2(t,x), & (t,x)\in Q,\\ u(0,x) = 0, & x\in\Omega\\ u(t,x) = 0, & (t,x)\in\Sigma\end{cases} \tag{5.5}$$

where

$$A(t,x) \equiv \{A^j(t,x)\}_{1\leq j\leq n} := A_1(t,x) - A_2(t,x),\quad \widetilde{q}(t,x) := \widetilde{q}_1(t,x) - \widetilde{q}_2(t,x) \text{ and}$$
$$q(t,x) := q_1(t,x) - q_2(t,x).$$

Using Green's formula, we have

$$\int_Q (\mathcal{L}_{A_1,q_1}u)(t,x)\overline{v}(t,x)\ \mathrm{d}x\mathrm{d}t - \int_Q u(t,x)\mathcal{L}_{-A_1,\overline{q}_1}\overline{v}(t,x)\ \mathrm{d}x\mathrm{d}t = \int_\Omega u(T,x)\overline{v}(T,x)\ \mathrm{d}x$$
$$-\int_\Omega u(0,x)\overline{v}(0,x)\ \mathrm{d}x - \int_\Sigma \overline{v}(t,x)\partial_\nu u(t,x)\ \mathrm{d}S_x\mathrm{d}t + \int_\Sigma u(t,x)\overline{\partial_\nu v(t,x)}\ \mathrm{d}S_x\mathrm{d}t,$$

which after using (5.5) becomes

$$2\int_Q (A(t,x)\cdot\nabla_x u_2(t,x))\overline{v}(t,x)\ \mathrm{d}x\mathrm{d}t + \int_Q \widetilde{q}(t,x)u_2(t,x)\overline{v}(t,x)\ \mathrm{d}x\mathrm{d}t = -\int_\Sigma \overline{v}(t,x)\partial_\nu u(t,x)\ \mathrm{d}S_x\mathrm{d}t. \tag{5.6}$$

We observe from (5.1)

$$A(t,x)\cdot\nabla_x u_2(t,x) = e^{\phi}\left(\lambda\omega\cdot A(t,x)B_2(t,x) + m(t,x)\right), \tag{5.7}$$

for some $m\in L^2(0,T;H^1(\Omega))$ satisfying the following estimate

$$\|m\|_{L^2(0,T;H^k(\Omega))} \leq C\lambda^k\delta^{-3}\langle\tau,\xi\rangle^3,\ \text{for } k\in\{0,1\} \text{ and } \lambda\geq\lambda_0. \tag{5.8}$$

An application of the Cauchy-Schwartz inequality together with (5.2),(5.3),(5.7) and (5.8) give us

$$\left(A\cdot\nabla_x u_2\right)(t,x)\overline{v}(t,x) = \left(\lambda(\omega\cdot A)B_2B + n\right)(t,x) \tag{5.9}$$

for some $n \in L^1(Q)$ satisfying

$$\|n\|_{L^1(Q)} \leq C\delta^{-6}\langle \tau, \xi\rangle^3. \tag{5.10}$$

Also from (5.1) and (5.2), it is clear that for $\lambda \geq \lambda_0$

$$\left|\int_Q \widetilde{q}(t,x)u_2(t,x)\overline{v}(t,x)\ \mathrm{d}x\mathrm{d}t\right| \leq C\delta^{-6}\langle \tau, \xi\rangle^3. \tag{5.11}$$

Now, we find an upper bound for the right hand side of (5.6) using the boundary Carleman estimate (3.1). We observe

$$\int_\Sigma \overline{v}(t,x)\partial_\nu u(t,x)\ \mathrm{d}S_x\mathrm{d}t = \int_{\Sigma_{-,\epsilon/2}(\omega_0)} \overline{v}(t,x)\partial_\nu u(t,x)\ \mathrm{d}S_x\mathrm{d}t + \int_{\Sigma_{+,\epsilon/2}(\omega_0)} \overline{v}(t,x)\partial_\nu u(t,x)\ \mathrm{d}S_x\mathrm{d}t$$

where $\Sigma_{+,\epsilon/2}(\omega_0)$ is the part of lateral boundary where we do not have any knowledge of Neumann measurements. Although the contribution from that part can be estimated by using the boundary Carleman estimate. Meanwhile for $\omega \in \mathbb{S}^{n-1}$ satisfying $|\omega-\omega_0| \leq \frac{\epsilon}{2}$, we have $\Sigma_{+,\epsilon/2}(\omega_0) \subseteq \Sigma_+(\omega)$ and $\Sigma_-(\omega) \subseteq \Sigma_{-,\epsilon/2}(\omega_0)$. Thus, we get from (5.4)

$$\|\overline{v}\partial_\nu u\|_{L^1\left(\Sigma_{-,\epsilon/2}(\omega_0)\right)} \leq Ce^{\beta\lambda}\delta^{-3}\left\|\partial_\nu u\right\|_{L^2\left(\Sigma_{-,\epsilon/2}(\omega_0)\right)} \leq Ce^{\beta\lambda}\delta^{-6}\langle\tau,\xi\rangle^3\|\Lambda_1-\Lambda_2\| \tag{5.12}$$

and

$$\begin{aligned}
\|\overline{v}\partial_\nu u\|_{L^1\left(\Sigma_{+,\epsilon/2}(\omega_0)\right)} &\leq C\delta^{-3}\|e^{-\phi}\partial_\nu u\|_{L^2\left(\Sigma_{+,\epsilon/2}(\omega_0)\right)} \leq \frac{2C\delta^{-3}}{\sqrt{\epsilon}}\sqrt{\int_{\Sigma_{+,\epsilon/2}(\omega_0)} e^{-2\phi}|\omega\cdot\nu(x)||\partial_\nu u|^2\ \mathrm{d}S_x} \\
&\leq \frac{2C\delta^{-3}}{\sqrt{\epsilon}}\sqrt{\int_{\Sigma_+(\omega)} e^{-2\phi}|\omega\cdot\nu(x)||\partial_\nu u|^2\ \mathrm{d}S_x} \\
&\leq \frac{2C\delta^{-3}}{\sqrt{\epsilon\lambda}}\left(\|e^{-\phi}\mathcal{L}_{A_1,q_1}(u)\|_{L^2(Q)} + \sqrt{\lambda}\|e^{-\phi}\partial_\nu u\|_{L^2(\Sigma_-(\omega))}\right).
\end{aligned}$$

Using (3.1), we have

$$\|\overline{v}\partial_\nu u\|_{L^1\left(\Sigma_{+,\epsilon/2}(\omega_0)\right)} \leq \frac{2C\delta^{-3}}{\sqrt{\epsilon\lambda}}\left(\left\|e^{-\phi}\left(2A\cdot\nabla_x u_2 + \widetilde{q}u_2\right)\right\|_{L^2(Q)} + \sqrt{\lambda}\|e^{-\phi}\partial_\nu u\|_{L^2\left(\Sigma_{-,\epsilon/2}(\omega_0)\right)}\right).$$

Finally using (5.3),(5.4),(5.7) and (5.8), we get

$$\|\overline{v}\partial_\nu u\|_{L^1\left(\Sigma_{+,\epsilon/2}(\omega_0)\right)} \leq C\delta^{-6}\langle\tau,\xi\rangle^3\left(\sqrt{\lambda} + e^{\beta\lambda}\|\Lambda_1-\Lambda_2\|\right). \tag{5.13}$$

After dividing the integral identity (5.6) by large λ and using Equations (5.7) to (5.13), we obtain

$$\left|\int_Q (\omega\cdot A)(t,x)B_2(t,x)B(t,x)\ \mathrm{d}x\mathrm{d}t\right| \leq C\left(\frac{1}{\sqrt{\lambda}} + e^{\beta\lambda}\|\Lambda_1-\Lambda_2\|\right)\delta^{-6}\langle\tau,\xi\rangle^3. \tag{5.14}$$

Next, we relate the integral in the left hand side of (5.14) with the Fourier-transform of A as is done in [43].

$$\begin{aligned}
&\int_Q (\omega\cdot A)(t,x)B_2(t,x)B(t,x)\ \mathrm{d}x\mathrm{d}t \\
&= \int_{\mathbb{R}^{1+n}} (\omega\cdot A)(t,x)\eta_\delta^2(t)e^{\left(-\int_0^\infty \omega\cdot A(t,x+s\omega)\ \mathrm{d}s\right)}\frac{\xi}{|\xi|}\cdot\nabla_x\left(e^{\int_{\mathbb{R}}\omega\cdot A(t,x+s\omega)\ \mathrm{d}s}e^{-i(t\tau+x\cdot\xi)}\right)\ \mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Now, using the decomposition $\mathbb{R}^n = \mathbb{R}\omega \oplus \omega^\perp$, we write $x := x_\perp + s\omega$ such that $x_\perp \in \omega^\perp$ and denote

$$f(s,t,x_\perp) := \int_s^\infty \omega \cdot A(t, x_\perp + \mu\omega)\ \mathrm{d}\mu.$$

Using these in the previous equation, we have

$$\begin{aligned}
&\int_Q (\omega \cdot A)(t,x) B_2(t,x) B(t,x)\ \mathrm{d}x\mathrm{d}t \\
&\quad = -\int_{\mathbb{R}} \eta_\delta^2(t) \int_{\omega^\perp} \frac{\xi}{|\xi|} \cdot \nabla_x \left(e^{\int_{\mathbb{R}} \omega\cdot A(t,x_\perp+s\omega)\ \mathrm{d}s} e^{-i(t\tau + x_\perp\cdot\xi)} \right) \left(\int_{\mathbb{R}} f'(s,t,x_\perp) e^{-f(s,t,x_\perp)}\ \mathrm{d}s \right) \mathrm{d}x_\perp \mathrm{d}t
\end{aligned}$$

where $\mathrm{d}x_\perp$ stands for the surface measure on $\omega^\perp$. Now using the Fundamental theorem of calculus and integration by parts, we get

$$\begin{aligned}
&\int_Q (\omega \cdot A)(t,x) B_2(t,x) B(t,x)\ \mathrm{d}x\mathrm{d}t \\
&= \int_{\mathbb{R}} \eta_\delta^2(t) \int_{\omega^\perp} \frac{\xi}{|\xi|} \cdot \nabla_x \left(e^{\int_{\mathbb{R}} \omega\cdot A(t,x_\perp+s\omega)\ ds} e^{-i(t\tau + x_\perp\cdot\xi)} \right) \left(e^{\left(-\int_{\mathbb{R}} \omega\cdot A(t,x_\perp+\mu\omega)\ d\mu\right)} - 1 \right) \mathrm{d}x_\perp \mathrm{d}t, \\
&= \int_{\mathbb{R}} \eta_\delta^2(t) \int_{\omega^\perp} e^{-i(t\tau + x_\perp\cdot\xi)} \frac{\xi}{|\xi|} \cdot \nabla_x \left(\int_{\mathbb{R}} \omega \cdot A(t, x_\perp + s\omega)\ \mathrm{d}s \right) \mathrm{d}x_\perp \mathrm{d}t \\
&= \int_{\mathbb{R}} \eta_\delta^2(t) \int_{\mathbb{R}^n} e^{-i(t\tau + x\cdot\xi)} \frac{\xi}{|\xi|} \cdot \nabla_x (\omega \cdot A)(t,x)\ \mathrm{d}x\mathrm{d}t = i|\xi| \int_{\mathbb{R}} \eta_\delta^2(t) \int_{\mathbb{R}^n} e^{-i(t\tau + x\cdot\xi)} \omega \cdot A(t,x)\ \mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Finally, we obtain

$$\int_Q (\omega \cdot A)(t,x) B_2(t,x) B(t,x)\ \mathrm{d}x\mathrm{d}t = i|\xi|\ \widehat{\eta_\delta^2 \omega \cdot A}(\tau, \xi), \tag{5.15}$$

where $\omega \cdot \xi = 0$.

Let us consider the spherical cap $\mathcal{C}_{\omega_0} := \{\omega \in \mathbb{S}^{n-1}; |\omega - \omega_0| < \frac{\epsilon}{2}\}$ and the set $\mathcal{H} = \cup_{\omega \in \mathcal{C}_{\omega_0}} \mathcal{H}_\omega$ where $\mathcal{H}_\omega$ is the plane passing through origin and perpendicular to ω. Now for $(\tau, \xi) \in \mathbb{R} \times \mathcal{H}$, $\lambda \geq \lambda_0$ and choosing $\omega(\xi) \in \mathcal{C}_{\omega_0}$ such that $\omega(\xi) \cdot \xi = 0$ in (5.14) and (5.15), we get

$$\left| \left(\eta_\delta^2 \partial_k \omega(\xi) \cdot A \right)^\wedge (\tau, \xi) \right| \leq C \left(\frac{1}{\sqrt{\lambda}} + e^{\beta\lambda} \|\Lambda_1 - \Lambda_2\| \right) \delta^{-6} \langle \tau, \xi \rangle^3 \tag{5.16}$$

where ∂_k denote the partial derivative with respect to the space variable x_k for $k \in \{1, 2, \cdots, n\}$. With the help of (5.16), we aim to establish the desired stability estimate via the Fourier inversion. Because of the stability result for analytic continuation in Proposition 5.3, it is enough to derive a uniform estimate for the Fourier transform of $\eta_\delta^2 A$ over an open cone only. This we do in the following lemma. We have crucially used the divergence free assumption on A to prove the following lemma.

Lemma 5.1. *If $div_x(A) = 0$ then for $(\tau, \xi) \in \mathbb{R} \times \mathcal{C}$ and $k \in \{1, 2, ..., n\}$ we have*

$$|\widehat{\eta_\delta^2 \partial_k A}(\tau, \xi)| \leq C \left(\frac{1}{\sqrt{\lambda}} + e^{\beta\lambda} \|\Lambda_1 - \Lambda_2\| \right) \delta^{-6} \langle \tau, \xi \rangle^3 \tag{5.17}$$

where, $\mathcal{C} \subseteq \mathcal{H}$ is an open cone in $\mathbb{R}^n$.

Proof. For a fixed nonzero $\xi \in \mathcal{H}$ and $k \in \{1, 2, ..., n\}$, we choose a set of $(n-1)$ linearly independent vectors from $\mathcal{C}_{\omega_0}$ which are perpendicular to ξ and denoted by $\{\omega_i(\xi)\}_{1 \le i \le n-1}$. Then, consider the following set of n linear equations

$$\sum_{j=1}^{n} \omega_i^j(\xi) \widehat{\eta_\delta^2 \partial_k A^j}(\tau, \xi) = G_i(\tau, \xi), \quad i \in \{1, 2, ..., n-1\}, \tag{5.18}$$

$$\text{and} \quad \sum_{j=1}^{n} \xi_j \widehat{\eta_\delta^2 \partial_k A^j}(\tau, \xi) = 0, \ \big(\text{since } \nabla_x \cdot A = 0 \text{ gives } \nabla_x \cdot (\eta_\delta^2 \, \partial_k A) = 0\big). \tag{5.19}$$

Here, $\{G_i(\tau, \xi)\}_{1 \le i \le n-1}$ are real numbers with a upper bound given by (5.17). Now, consider the matrix M_ξ related to the system of equations (5.18) and (5.19) as follow

$$M_\xi = \begin{pmatrix} \omega_1^1(\xi) & \omega_1^2(\xi) & \cdots & \omega_1^n(\xi) \\ \omega_2^1(\xi) & \omega_2^2(\xi) & \cdots & \omega_2^n(\xi) \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{n-1}^1(\xi) & \omega_{n-1}^2(\xi) & \cdots & \omega_{n-1}^n(\xi) \\ \frac{\xi_1}{|\xi|} & \frac{\xi_2}{|\xi|} & \cdots & \frac{\xi_n}{|\xi|} \end{pmatrix}.$$

From our assumption of $\{\omega_i(\xi)\}_{1 \le i \le n-1}$, it is clear that M_ξ is a non-singular matrix which is homogeneous of order zero in ξ. Hence it suffices to restrict the discussion for $\xi \in \mathbb{S}^{n-1}$. Thus we take some $\xi_0 \in \mathbb{S}^{n-1} \cap \mathcal{H}$ and see $0 < |\det M_{\xi_0}|$. Now the continuity of determinants readily implies that for some neighborhood of ξ_0 in $\mathbb{S}^{n-1}$ say $\widetilde{\mathcal{C}}$ we have

$$0 < c \le |\det M_\xi| \tag{5.20}$$

since the plane $\mathcal{H}_\xi$ changes continuously as we vary $\xi \in \widetilde{\mathcal{C}}$, giving a linearly independent set of vectors $\{\omega_i(\xi)\}_{1 \le k \le n-1}$ which also depend continuously on ξ. Note here, the constant $c > 0$ in (5.20) is independent of $\xi \in \widetilde{\mathcal{C}}$. Now for any $r > 0$ and $(\tau, \xi) \in \mathbb{R} \times \widetilde{\mathcal{C}}$, using the uniform positive lower bound for the matrix M_ξ in the system of linear equations (5.18) and (5.19), we get

$$|\widehat{\eta_\delta^2 \partial_k A^j}(r\tau, r\xi)| \le C \left(\frac{1}{\sqrt{\lambda}} + e^{\beta\lambda} \|\Lambda_1 - \Lambda_2\| \right) \delta^{-6} \langle r\tau, r\xi \rangle^3 \tag{5.21}$$

where $k, j \in \{1, 2, \cdots, n\}$. Define $\mathcal{C} \equiv \cup_{r>0} r\widetilde{\mathcal{C}}$, which is an open cone in $\mathbb{R}^n$. Thus, for $(\tau, \xi) \in \mathbb{R} \times \mathcal{C}$, we get (5.17). $\square$

Lemma 5.2. *For $R \ge 1$ and $\delta \in (0, T/4)$, there exist $C > 0$ and $\theta \in (0, 1)$ such that the following estimate holds*

$$\left\| |\xi| \widehat{\eta_\delta^2 A} \right\|_{L^\infty(B(0,R))} \le C e^{R(1-\theta)} \left(\frac{1}{\sqrt{\lambda}} + e^{\beta\lambda} \|\Lambda_1 - \Lambda_2\| \right)^\theta \delta^{-6\theta} R^{3\theta}. \tag{5.22}$$

Proof. Fix $k \in \{1, 2, \cdots, n\}$ and consider the analytic function $f_{R,k}$ given by

$$f_{R,k}(t, x) = \widehat{\eta_\delta^2 \partial_k A}(Rt, Rx), \qquad \text{for } R > 0 \text{ and } (t, x) \in \mathbb{R}^{n+1}.$$

For any multi-index γ in $\mathbb{R}^{1+n}$ we observe

$$\begin{aligned}\left|\partial^\gamma_{(t,x)} f_{R,k}(t,x)\right| = \left|\partial^\gamma_{(t,x)} \widehat{\eta_\delta^2 \partial_k A}(Rt,Rx)\right| &= \left|\int_{\mathbb{R}^{1+n}} e^{-iR(s,y)\cdot(t,x)}(-i)^{|\gamma|}R^{|\gamma|}(s,y)^\gamma(\eta_\delta^2\partial_k A)(s,y)\ \mathrm{d}s\mathrm{d}y\right| \\ &\leq \int_{\mathbb{R}^{1+n}} R^{|\gamma|}\left(s^2+|y|^2\right)^{\frac{|\gamma|}{2}}(\eta_\delta^2\partial_k A)(s,y)\ \mathrm{d}s\mathrm{d}y, \\ &\leq (2T^2)^{\frac{|\gamma|}{2}} R^{|\gamma|}\int_{\mathbb{R}^{1+n}} |(\eta_\delta^2\partial_k A)(s,y)|\ \mathrm{d}s\mathrm{d}y.\end{aligned}$$

Now using $\operatorname{diam}(\Omega) < T$ and apriori bound of A, we have

$$\left|\partial^\gamma_{(t,x)} f_{R,k}(t,x)\right| \leq C_*\ (2T^2)^{\frac{|\gamma|}{2}}\ R^{|\gamma|} = C_*(\sqrt{2}T)^{|\gamma|}|\gamma|!\ \frac{R^{|\gamma|}}{|\gamma|!},$$

which immediately gives

$$\left|\partial^\gamma_{(t,x)} f_{R,k}(t,x)\right| \leq C_* e^R \frac{|\gamma|!}{(T^{-1})^{|\gamma|}}, \quad \text{for } (t,x)\in\mathbb{R}^{n+1} \text{ and multi-index } \gamma. \tag{5.23}$$

Let us recall a specific variant of analytic continuation results from [5] which states

Proposition 5.3 (Appendix A of [5]). *For a real analytic function f in $B(0,2)\subset\mathbb{R}^d, d\geq 2$ satisfying*

$$\|\partial^\gamma f\|_{L^\infty(B(0,2))} \leq \frac{M|\gamma|!}{(2\rho)^{|\gamma|}}, \quad \forall\gamma\in(\mathbb{N}\cup\{0\})^d$$

we have

$$\|f\|_{L^\infty(B(0,1))} \leq N_\rho M^{1-\theta}\|f\|^\theta_{L^\infty(U)}$$

where $M,\rho,N_\rho>0$ and U is an non-empty open set in $B(0,1)$. Moreover, $\theta\in(0,1)$ depends only on d,ρ and $|U|$.

For related results on analytic continuation we suggest the reader to consult [49] and also [57]. Now we appeal to Proposition 5.3 for the function $f_{R,k}$ where we take $(\mathbb{R}\times\mathcal{C})\cap B(0,1)$ as U. Since $f_{R,k}$ satisfies (5.23), therefore using Proposition 5.3, we obtain

$$\|f_{R,k}\|_{L^\infty(B(0,1))} \leq Ce^{R(1-\theta)}\ \|f_{R,k}\|^\theta_{L^\infty((\mathbb{R}\times\mathcal{C})\cap B(0,1))}, \quad \text{for some } \theta\in(0,1). \tag{5.24}$$

Since $\|f_{R,k}\|_{L^\infty(B(0,1))} = \left\|\widehat{\eta_\delta^2\partial_k A}\right\|_{L^\infty(B(0,R))}$, therefore using lemma (5.1) and equation (5.24), we get

$$\left\|\widehat{\eta_\delta^2\partial_k A}\right\|_{L^\infty(B(0,R))} \leq Ce^{R(1-\theta)}\left(\frac{1}{\sqrt{\lambda}} + e^{\beta\lambda}\|\Lambda_1-\Lambda_2\|\right)^\theta \delta^{-6\theta}(1+R^2)^{\frac{3\theta}{2}}.$$

which can be expressed for $R\geq 1$ in the following form

$$\left\||\xi|\widehat{\eta_\delta^2 A}\right\|_{L^\infty(B(0,R))} \leq Ce^{R(1-\theta)}\left(\frac{1}{\sqrt{\lambda}} + e^{\beta\lambda}\|\Lambda_1-\Lambda_2\|\right)^\theta \delta^{-6\theta}R^{3\theta}.$$

□

Now combining (5.22) and the apriori assumption on the potentials, we establish a Sobolev bound of A in terms of the partial DN map. The main argument here is to set a comparison between the large parameters λ and R. Also, we have to choose the small parameter δ accordingly. But first we observe the following estimate

$$\begin{aligned}
\|\eta_\delta^2 A\|_{L^2(Q)}^{\frac{2}{\theta}} &= \left(\int_{\mathbb{R}^{1+n}} |\widehat{\eta_\delta^2 A}|^2(s,y)\ \mathrm{d}s\mathrm{d}y\right)^{\frac{1}{\theta}} \\
&= \left(\int_{B(0,R)} |\widehat{\eta_\delta^2 A}|^2(s,y)\ \mathrm{d}s\mathrm{d}y + \int_{B(0,R)^c} |\widehat{\eta_\delta^2 A}|^2(s,y)\ \mathrm{d}s\mathrm{d}y\right)^{\frac{1}{\theta}} \\
&\leq 2^{\frac{1}{\theta}-1}\left(\left(\underbrace{\int_{B(0,R)} |\widehat{\eta_\delta^2 A}|^2(s,y)\ \mathrm{d}s\mathrm{d}y}_{T_1}\right)^{\frac{1}{\theta}} + \left(\underbrace{\int_{B(0,R)^c} |\widehat{\eta_\delta^2 A}|^2(s,y)\ \mathrm{d}s\mathrm{d}y}_{T_2}\right)^{\frac{1}{\theta}}\right)
\end{aligned} \tag{5.25}$$

where the last line uses the convexity of $f(x) = x_+^{1/\theta};\ \theta \in (0,1)$. Now T_2 can be easily estimated after using the apriori assumptions for the potentials. We see

$$\begin{aligned}
T_2 &= \int_{B(0,R)^c} |\widehat{\eta_\delta^2 A}|^2(s,y)\ \mathrm{d}s\mathrm{d}y \\
&\leq \frac{1}{R^2}\int_{\mathbb{R}^{1+n}} \langle \tau,\xi\rangle^2 |\widehat{\eta_\delta^2 A}|^2(s,y)\ \mathrm{d}s\mathrm{d}y \leq \frac{1}{R^2}\|\eta_\delta^2 A\|_{H^1(Q)}^2 \leq \frac{C}{\delta^2 R^2}.
\end{aligned} \tag{5.26}$$

To estimate T_1, we use lemma 5.2 . We break T_1 into two parts. We consider $T_1 = T_{11} + T_{12}$, where

$$\begin{aligned}
T_{11} &:= \int_{B(0,R)\cap\{(s,y);\ |y|\leq R^{-\frac{3}{n}}\}} |\widehat{\eta_\delta^2 A}|^2(s,y)\ \mathrm{d}s\mathrm{d}y \\
&\leq \|\widehat{\eta_\delta^2 A}\|_{L^\infty(\mathbb{R}^{n+1})}\int_{-R}^{R}\int_{|y|\leq R^{-\frac{3}{n}}}\ \mathrm{d}s\mathrm{d}y \leq CR^{-2}
\end{aligned} \tag{5.27}$$

and

$$\begin{aligned}
T_{12} &:= \int_{B(0,R)\cap\left\{(s,y);\ |y|> R^{-\frac{3}{n}}\right\}} |\widehat{\eta_\delta^2 A}|^2(s,y)\ \mathrm{d}s\mathrm{d}y \\
&\leq e^{2R(1-\theta)}\left(\frac{1}{\sqrt{\lambda}} + e^{\beta\lambda}\|\Lambda_1-\Lambda_2\|\right)^{2\theta}\delta^{-12\theta}R^{6\theta+n+1+\frac{6}{n}}.
\end{aligned} \tag{5.28}$$

Because of the support condition of $\{\eta_\delta\}_{\delta>0}$, we have

$$\|A\|_{L^2(Q)}^2 \leq \|\eta_\delta^2 A\|_{L^2(Q)}^2 + C\delta. \tag{5.29}$$

Taking $\alpha = 6 + \frac{n^2+n+6}{n\theta}$ and combining (5.25)-(5.28), we obtain

$$\begin{aligned}
\|A\|_{L^2(Q)}^{\frac{2}{\theta}} \leq& C\left(\frac{R^\alpha}{\delta^{12}}e^{\frac{2R(1-\theta)}{\theta}}\left(\frac{1}{\lambda} + e^{2\beta\lambda}\|\Lambda_1-\Lambda_2\|^2\right) + \frac{1}{\delta^{\frac{2}{\theta}}R^{\frac{2}{\theta}}} + \delta^{\frac{1}{\theta}}\right) \\
\leq& C\left(\underbrace{\frac{R^\alpha e^{\frac{2R(1-\theta)}{\theta}}}{\lambda\delta^{12}}}_{I} + \underbrace{\frac{R^{\alpha\theta}e^{\frac{2R(1-\theta)}{\theta}+2\beta\lambda}}{\delta^{12}}\|\Lambda_1-\Lambda_2\|^2}_{II} + \underbrace{\frac{1}{\delta^{\frac{2}{\theta}}R^{\frac{2}{\theta}}}}_{III} + \underbrace{\delta^{\frac{1}{\theta}}}_{IV}\right).
\end{aligned} \tag{5.30}$$

We now choose λ, δ and R in a way so that the terms (I), (III) and (IV) in (5.30) are comparable. That is when

$$\delta = \frac{1}{R^{\frac{2}{3}}} \quad \text{and} \quad \lambda = R^{\alpha+8+\frac{2}{3\theta}} e^{\frac{2R(1-\theta)}{\theta}} \tag{5.31}$$

and hence there exists $\kappa > 0$ (independent of R) such that II of (5.30) can be bounded by

$$e^{e^{\kappa R}} \|\Lambda_1 - \Lambda_2\|^2. \tag{5.32}$$

Combining (5.30)-(5.32), it is clear that

$$\|A\|_{L^2(Q)}^{\frac{2}{\theta}} \leq C \left(\frac{1}{R^{\frac{2}{3\theta}}} + e^{e^{\kappa R}} \|\Lambda_1 - \Lambda_2\|^2 \right). \tag{5.33}$$

We now choose $R > 0$ large enough (which in turn depends on the smallness of partial DN map) which is $R = \frac{1}{\kappa} \log \left| \log \|\Lambda_1 - \Lambda_2\| \right|$, so that (5.33) becomes

$$\|A\|_{L^2(Q)}^{\frac{2}{\theta}} \leq C \left(\|\Lambda_1 - \Lambda_2\| + (\log|\log\|\Lambda_1 - \Lambda_2\||)^{-\frac{2}{3\theta}} \right). \tag{5.34}$$

We note that, (5.34) can be easily derived for the case when $\|\Lambda_1 - \Lambda_2\|$ is not so small. This concludes the proof for stability of first order coefficients from the partial DN map.

Now we establish the stability result for the zeroth order term. There will be no zeroth order term left in (5.6) once we divide it by large λ. So we have to make necessary changes for deriving Fourier estimates. We explicitly use here the stability result for the first order terms (5.34). We consider a different exponentially growing solutions for $\mathcal{L}_{A_2, q_2}$, whereas the geometric optics for $\mathcal{L}_{-A_1, \overline{q}_1}$ is same as before which is (5.2). We have

$$u_2(t,x) = e^{\phi(t,x)} \left(B_2 + R_2 \right)(t,x)$$

where

$$B_2(t,x) = e^{-i(t\tau + x\cdot\xi)} \eta_\delta(t) e^{\left(\int_0^\infty \omega\cdot A_2(t, x+s\omega)\, \mathrm{d}s\right)},$$

and $R_2 \in L^2(0,T; H^2(\Omega))$ satisfying for $k \in \{0,1,2\}$

$$\|R_2\|_{L^2(0,T;H^k_\lambda(\Omega))} \leq C\lambda^{-1+k}\delta^{-3}\langle \tau, \xi\rangle^3. \tag{5.35}$$

For convenience, we rewrite the integral inequality

$$2\int_Q (A\cdot\nabla_x u_2)(t,x)\overline{v}(t,x)\, \mathrm{d}x\mathrm{d}t + \int_Q \widetilde{q}(t,x) u_2(t,x)\overline{v}(t,x)\, \mathrm{d}x\mathrm{d}t = \int_\Sigma \overline{v}(t,x)\partial_\nu u(t,x)\, \mathrm{d}S_x\mathrm{d}t. \tag{5.36}$$

First, we simplify all the terms present in left hand side of (5.36). We observe

$$\widetilde{q}(t,x)u_2(t,x)\overline{v}(t,x) = \widetilde{q} e^{-it\tau - ix\cdot\xi} \eta_\delta^2(t) e^{-\int_0^\infty \omega\cdot A(t,x+s\omega)\mathrm{d}s} + B_2(t,x)R(t,x) + B(t,x)R_2(t,x). \tag{5.37}$$

We use Cauchy-Schwarz inequality alongwith (5.3) and (5.35) to obtain

$$\|B_2 R\|_{L^1(Q)} + \|BR_2\|_{L^1(Q)} \leq \frac{C}{\lambda}\delta^{-3}\langle\tau,\xi\rangle^3 \tag{5.38}$$

The other term present in the L.H.S of (5.36) is

$$2(A\cdot\nabla_x u_2)(t,x)\overline{v}(t,x) = 2\left(\lambda\omega\cdot AB_2 + \lambda\omega\cdot AR_2 + A\cdot\nabla_x B_2 + A\cdot\nabla_x R_2\right)(t,x)\left(\overline{B} + \overline{R}\right)(t,x) \tag{5.39}$$

which is estimated by using the remainder term estimates given in (5.3) and (5.35)

$$\left|\int_Q (A\cdot\nabla_x u_2)(t,x)\overline{v}(t,x)\ \mathrm{d}x\mathrm{d}t\right| \leq C\lambda\|A\|_{L^2(Q)}\delta^{-3}\langle\tau,\xi\rangle^3. \tag{5.40}$$

To estimate the boundary term in the R.H.S of (5.36), we proceed as before and obtain

$$\left|\int_\Sigma \overline{v}(t,x)\partial_\nu u(t,x)\ \mathrm{d}S_x\mathrm{d}t\right| \leq C\left(\sqrt{\frac{\mathcal{K}}{\lambda}}+\delta^{-6}\langle\tau,\xi\rangle^3 e^{\beta\lambda}\|\Lambda_1-\Lambda_2\|\right) \tag{5.41}$$

Here $\mathcal{K}$ is the R.H.S of the boundary Carleman estimate (3.1) applied to $\mathcal{L}_{A_1,q_1}$ for u,

$$\mathcal{K}=\int_Q e^{-2\phi}|\mathcal{L}_{A_1,q_1}u|^2\ \mathrm{d}x\mathrm{d}t+\lambda\int_{\Sigma_-(\omega)} e^{-2\phi}|\omega\cdot\nu(x)||\partial_\nu u|^2\ \mathrm{d}S_x\mathrm{d}t. \tag{5.42}$$

From (5.5) we have, $\mathcal{L}_{A_1,q_1}u=2A\cdot\nabla_x u_2+\tilde{q}u_2$. Hence we see

$$e^{-\phi}\mathcal{L}_{A_1,q_1}u=2\left(A\cdot\nabla_x B_2+\lambda\omega\cdot AR_2+A\cdot\nabla_x B_2+A\cdot\nabla_x R_2\right)+\tilde{q}(B_2+R_2). \tag{5.43}$$

Thus, we use (5.35) to obtain

$$\mathcal{K}\leq C\left(\lambda^2\|A\|^2_{L^2(Q)}+1+e^{\beta\lambda}\|\Lambda_1-\Lambda_2\|^2\right)\delta^{-12}\langle\tau,\xi\rangle^6. \tag{5.44}$$

Combining (5.36)-(5.44), we conclude for $(\tau,\xi)\in\mathbb{R}\times\mathcal{H}$

$$\begin{aligned}\left|\widehat{\eta_\delta^2\widetilde{q}}(\tau,\xi)\right| &= \left|\int_Q e^{-i(t\tau+x\cdot\xi)}\eta_\delta^2(t)\widetilde{q}(t,x)\ \mathrm{d}x\mathrm{d}t\right| \\ &\leq C\left(\lambda\|A\|_{L^2(Q)}+\frac{1}{\sqrt{\lambda}}+e^{\beta\lambda}\|\Lambda_1-\Lambda_2\|\right)\delta^{-6}\langle\tau,\xi\rangle^3.\end{aligned} \tag{5.45}$$

Basically (5.45) gives estimate for the Fourier transform of $\eta_\delta^2\widetilde{q}$ over the cone $\mathbb{R}\times\mathcal{H}$ in $\mathbb{R}^{1+n}$. So we apply the stability result for analytic continuation as done before to obtain similar estimate over arbitrary large balls. Mimicing the arguments presented before, we get the following estimate similar to (5.30)

$$\begin{aligned}\|\widetilde{q}\|^{\frac{2}{\theta}}_{L^2(Q)} &\leq C\left(\frac{R^{\alpha'}}{\delta^{12}}e^{\frac{2R(1-\theta)}{\theta}}\left(\lambda^2\|A\|^2_{L^2(Q)}+1+e^{\beta\lambda}\|\Lambda_1-\Lambda_2\|^2\right)+\frac{1}{\delta^{\frac{2}{\theta}}R^{\frac{2}{\theta}}}+\delta^{\frac{1}{\theta}}\right) \\ &\leq C\left(\underbrace{\frac{R^{\alpha'}e^{\frac{2R(1-\theta)}{\theta}}}{\delta^{12}}\lambda^2\|A\|^2_{L^2(Q)}}_{I}+\underbrace{\frac{R^{\alpha'}e^{\frac{2R(1-\theta)}{\theta}}}{\delta^{12}}}_{II}+\underbrace{\frac{R^{\alpha\theta}e^{\frac{2R(1-\theta)}{\theta}+\beta\lambda}}{\delta^{12}}\|\Lambda_1-\Lambda_2\|^2}_{III}+\underbrace{\frac{1}{\delta^{\frac{2}{\theta}}R^{\frac{2}{\theta}}}}_{IV}+\underbrace{\delta^{\frac{1}{\theta}}}_{V}\right).\end{aligned} \tag{5.46}$$

We choose δ and R such that (II),(IV) and (V) of (5.46) are comparable. That is when

$$\delta=\frac{1}{R^{\frac{2}{3}}}\ \text{and}\ \lambda=R^{\alpha'+8+\frac{2}{3\theta}}e^{\frac{2R(1-\theta)}{\theta}}$$

where $\alpha'=6+\frac{n+1}{\theta}$. Now using the stability result (5.34), we obtain

$$\|\widetilde{q}\|^{\frac{2}{\theta}}_{L^2(Q)}\leq C\left(e^{\kappa R}\|\Lambda_1-\Lambda_2\|^{2\mu_1}+e^{\kappa R}\left|\log|\log\|\Lambda_1-\Lambda_2\||\right|^{-2\mu_2}+e^{e^{\kappa R}}\|\Lambda_1-\Lambda_2\|^2+\frac{1}{R^{\frac{2}{3\theta}}}\right) \tag{5.47}$$

for some constants $\kappa > 0$ and $\mu_1, \mu_2 > 0$. Taking $R = \frac{\mu_1}{\kappa} \log\log|\log\|\Lambda_1 - \lambda_2\||$, we get from (5.47) that $\|\widetilde{q}\|_{L^2(Q)}^{\frac{2}{\theta}}$ has the upper bound

$$\begin{aligned} &\|\Lambda_1 - \Lambda_2\|^{2\mu_1} \left(\log|\log\|\Lambda_1 - \Lambda_2\||\right)^{-2\mu_2} + \left(\log|\log\|\Lambda_1 - \Lambda_2\||\right)^{-\mu_2} \\ &\quad + \|\Lambda_1 - \Lambda_2\|^2 |\log\|\Lambda_1 - \Lambda_2\|| + \left(\log\log|\log\|\Lambda_1 - \Lambda_2\||\right)^{-\frac{2}{3\theta}}. \end{aligned} \tag{5.48}$$

We note that our choice of R related to the smallness of δ and largeness of λ. Hence, the estimate (5.48) is valid only when $\|\Lambda_1 - \Lambda_2\|$ is small enough. The other case follows easily. Also, we need smallness of $\|\Lambda_1 - \Lambda_2\|$ such that the following hold

$$\|\Lambda_1 - \Lambda_2\|^{\mu_1} \left(\log|\log\|\Lambda_1 - \Lambda_2\||\right)^{-2\mu_2} + \|\Lambda_1 - \Lambda_2\| |\log\|\Lambda_1 - \Lambda_2\|| \le C.$$

Thus for both the cases, we arrive at the following estimate where C, α_1 and $\alpha_2 > 0$

$$\|\widetilde{q}\|_{L^2(Q)} \le C \left(\|\Lambda_1 - \Lambda_2\|_{L^2(Q)}^{\alpha_1} + |\log|\log|\log\|\Lambda_1 - \lambda_2\|||^{-\alpha_2} \right). \tag{5.49}$$

Now we want to prove the stability result for $q := q_1 - q_2$. We recall

$$q(t, x) = \widetilde{q}(t, x) + \nabla_x \cdot A(t, x) + \left(|A_1|^2 - |A_2|^2\right)(t, x).$$

Hence, we obtain the following

$$\|q\|_{L^2(Q)} \le \|\widetilde{q}\|_{L^2(Q)} + (2m + 1)\|A\|_{L^2(0,T;H^1(\Omega))}. \tag{5.50}$$

Since our assumptions on the first order perturbations are more than H^1, we can translate the L^2 norm estimates to that of H^1 using logarithmic convexity for Sobolev norms. Thus there exist $C > 0$ and $\theta \in (0, 1)$ depending only on m and Q so that we have

$$\|A\|_{H^1(Q)} \le C\|A\|_{L^2(Q)}^{\theta} \tag{5.51}$$

Using (5.51), the L^2 stability results in (5.34) and (5.49) in (5.50), we obtain

$$\|q\|_{L^2(Q)} \le C \left(\|\Lambda_1 - \Lambda_2\|^{\beta_1} + |\log|\log|\log\|\Lambda_1 - \Lambda_2\|||^{-\beta_2} \right).$$

for some C, β_1 and $\beta_2 > 0$. This completes the proof of Theorem 2.1.

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